Basics on commutative algebra

August 2014

1 Modules

Rings are supposed to be commutative and unitary.

Definition 1.1. *1. Let A be a ring, an A-module M is a commutative group M endowed with a product*

$$\begin{array}{ccc} A \times M & \to & M \\ (a,x) & \mapsto & a * x \end{array}$$

such that

(a) associativity:

$$a,b \in A, x \in M, a * (b * x) = (ab) * x$$

(b) distributivity:

$$(a+b) * x = a * x + b * y, y \in M$$

(c) $1_A * x = x$ ($\Rightarrow 0_A * x = 0_M$, $(-1)_A * x = -x$)

2. We define the expression ax for $a \in A$ and $x \in M$ as

$$ax \equiv a * x$$

3. If $f : M \to N$, (M, N are A-modules) is a group homomorphism and $f(a * x) = a * f(x)$, $\forall a \in A$, $\forall x \in M$, then f is called a module homomorphism

4. A submodule of M is a subgroup N such that $ax \in N$, $\forall x \in N$, $\forall a \in A$

5. let $S \subset M$ subset, a submodule of M generated by S is the smallest submodule of M containing S. It is also equal to

$$\bigcap_{N \text{ submodule containing } S} N = \{a_1x_1 + ...a_nx_n | a_1, ..., a_n \in A,\ x_1, ..., x_n \in S, n \text{ varies}\}$$

Exercise 1.2. *All vector spaces are modules. Every ring is a module over itself, and its submodules are precisely its ideals. Modules over $\mathbb{Z}$ are exactly the abelian groups up to isomorphism.*

Example 1.3. *$f : M \to N$ linear map, $ker(f)$ submodule of M and $Im(f)$ submodule of N.*

Exercise 1.4. *$f : M \to N$ linear is injective $\Leftrightarrow$ $ker(f) = \{0\}$*

Definition 1.5. 1. *Let S be a subset of M. S is free if $\forall x_1, ..., x_n \in S$, $\forall a_1, ..., a_n \in A$, $a_1x_1 + ..., a_nx_n = 0 \implies a_1 = ... = a_n = 0$;*

2. *A basis of M is a free subset which generates M;*

3. *$(M_i)_{i\in I}$ family of A-modules*

$$\oplus_{i\in I} M_i = \{(a_i)_{i\in I}, a_i = 0, \forall i \in I except\ for\ finitely\ many\ of\ them\} \subset \prod_{i\in I} M_i$$

If $M_i = A$ $\forall i \in I$ then we denote $\oplus_{i\in I} M_i = A^{(I)}$.

4. *M is a free module if there is a basis ($\Leftrightarrow$ there is an isomorphism $A^{(I)} \equiv M$ for some set I, $A^{(I)} = \oplus_I A \neq A^I$).*

   Exercise 1.6. *Prove the above equivalence.*

5. *Let N be a submodule of M. M/N is the quotient as group endowed with a natural structure of module by $a * \bar{x} = \overline{ax}$, $\forall a \in A$, $\forall x \in M$, $\bar{x}$ is the class of x mod N. M/N is a A-module, and the canonical $M \to M/N$ is linear*

Example 1.7. 1. *$n \geqslant 1$, $\mathbb{Z}/n\mathbb{Z}$ $\mathbb{Z}$-module is not free $\forall x \in \mathbb{Z}/n\mathbb{Z}$, $n \cdot x = 0$*

2. *$\mathbb{Q}$ $\mathbb{Z}$-module is not free. If $r \neq 0$, $n \cdot r = 0 \Rightarrow n = 0$, and if $r_1 = p_1/q_1$, $r_2 = p_2/q_2$; $p_i, q_i \in \mathbb{Z}$, $q_i \neq 0$.*

$$(q_1p_2)r_1 - (q_2p_1)r_2 = 0$$

If S is a basis of $\mathbb{Q}$ as $\mathbb{Z}$-module $\Rightarrow S = \{r_0\}$ but $\mathbb{Q} = r_0\mathbb{Z}$, $1/2r_0 \notin r_0\mathbb{Z}$.

Exercise 1.8. *If M is finitely generated and has a basis, then the basis is finite.*

Exercise 1.9. *If a finitely generated $\mathbb{Z}$-module has two different bases, they are the same size.*

Exercise 1.10. *(ISOMORPHISM THEOREM) If $f : M \to N$ linear surjective map, prove that $M/ker(f)$ is isomorphic to N as modules.*

1.1 Noetherian modules:

Definition 1.11. 1. *M is A-module is noetherian if every submodule of M is finitely generated (generated by a finite subset).*

2. *A is noetherian ring if it is noetherian as an A-module over itself. This is equivalent to saying that the ideals of A are finitely generated.*

Exercise 1.12. *A noetherian, M A-module then M is noetherian if and only if M is finitely generated.*

Example 1.13. 1. *Fields are noetherian, $\mathbb{Z}$ is noetherian;*

2. **Theorem (Hilbert)** *A is noetherian $\Rightarrow$ $A[X]$ is noetherian.*

3. *k field, $k[X_1, ..., X_n]$ is noetherian.*
4. *If A is noetherian then A/I is noetherian $\forall I$ ideal of A.*

Exercise 1.14. *Prove the converse of Hilbert's Theorem: If $A[X]$ is noetherian $\Rightarrow A$ is noetherian.*

1.2 Localization

Definition 1.15. *A is a ring. A multiplicative subset S of A is a set such that:*

1. *$1 \in S$,*
2. *$\forall s, t \in S$, $st \in S$,*

Definition 1.16. *$S^{-1}A$ is the localization of A with respect to S. It is defined as the set of equivalence classes $\{a/s, a \in A, s \in S\}$, where equivalence is defined below*

$a/s = b/t \Leftrightarrow (at - bs)s' = 0$, for some $s' \in S$;

$a/s + b/t = (at + bs)/(st)$.

$a/s \cdot b/t = (ab)/(st)$.

Example 1.17. 1. *$a/s = a'/s' \Rightarrow (as' - a's)s'' = 0$;*

2. *$(at + bs)/(st) = (a't + bs')/(s't)$*

   $((at + bs)s't - (a't + bs')st = as't^2 + bss't - a'st^2 - bss't = (as - a's)t^2$ *is killed by* s''

We can check that this defines a structure of commutative unitary ring on $S^{-1}A$, the map $A \to S^{-1}A$ sending a to $a/1$ is a ring homomorphism.

Remarque 1.18. *if A is an integral domain, $S = A - \{0\}$ is multiplicative subset of A. $S^{-1}A$ is a field of fraction of A. If A integral, then $\forall T$ multiplicative subset of A, $T^{-1}A \subset Frac(A)$.*

Example 1.19. *$A = \mathbb{Q}[X, Y]/(XY) = \mathbb{Q}[x, y]$, $S = \{x^n | n \geqslant 1\} \cup \{1\}$ and $S^{-1}A = \mathbb{Q}[x, 1/x] = \{P(x)/x^n | P(x) \in \mathbb{Q}[x], n \geqslant 0\}$.*

Example 1.20. 1. *Let $f \in A$, $A_f := S^{-1}A$, $S = \{f^n | n \geqslant 1\} \cup \{1\}$;*

2. *If f is nilpotent $\Rightarrow 0 \in S$, $\Rightarrow 0 \in S^{-1}A$ is invertible in $S^{-1}A \Rightarrow S^{-1}A = \{0\}$;*
3. *Let $\mathfrak{p}$ be a prime ideal of A, $A_\mathfrak{p} = S^{-1}A$, $S = A - \mathfrak{p}$ multiplicative and δ defined as*

$$\begin{array}{cccc} \delta: & A & \to & S^{-1}A \\ & a & \mapsto & a/1 \end{array}$$

   of A. We have the following universal property: Let $f : A \to B$ be a ring homomorphism then f factorizes though $S^{-1}A$ if and only if $f(S) \subset B^\times$ (invertible)

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "\delta"'] & B \\ S^{-1}A \arrow[ur, "\tilde{f}"'] & \end{tikzcd}$$

$\tilde{f}$ is a ring homomorphism $\tilde{f}(a/s) = f(a)f(s)^{-1}$, $f(s) \in B$.

Exercise 1.21. *Prove the above universal property.*

Notation: B ring, $Spec(B) = \{prime\ ideals\ of\ B\}$ (Spectrum of B). If $f : A \to B$ ring homomorphism $\Rightarrow$

$$\begin{array}{llll} Spec(f): & Spec(B) & \to & Spec(A) \\ & Q & \mapsto & f^{-1}(Q) \end{array}$$

Proposition 1.22. *$Spec(f) : Spec(S^{-1}A) \to Spec(A)$ induces a bijection from $Spec(S^{-1}A)$ to $\{Q \in Spec(A) | Q \cap S = \emptyset\}$*

Proof. Exercise □

M A-module, S multi subset of $A \to S^{-1}A$. Let's define $S^{-1}M = \{x/s | x \in M, s \in S\}$

1. $x/s = y/t \Leftrightarrow s'(tx - sy) = 0$, for some $s' \in S$;
2. $x/s + y/t = (tx + sy)/(st)$;
3. $a/s \cdot x/t = (ax)/(st)$.

Exercise 1.23. *Prove the above definitions are well defined and do make $S^{-1}M$ into a $S^{-1}A$-module.*

Exercise 1.24. *Let S be a multiplicatively closed subset of a ring A and let M be a finitely generated A-module. Prove that $S^{-1}M = 0$ if and only if there exists $s \in S$ such that $sM = \{0\}$.*

1.3 Tensor product

A ring; M, N A-modules;

Definition 1.25. *The tensor product of M, N over A. Let H be a A-module endowed with a bilinear map $\delta : M \times N \to H$ with the universal property for every bilinear map, $f : M \times N \to L$ there exists a unique factorization*

$$\begin{tikzcd} M \times N \arrow[r, "f"] \arrow[d] & L \\ H \arrow[ur, "\tilde{f}"'] & \end{tikzcd}$$

with $\tilde{f}$ linear map .

Proposition 1.26. *The tensor exists and is unique.*

Proof. 1. **Uniqueness:** Exercise.

2. **Existence:**
$A^{(M \times N)}$ $(x, y) \in M \times N$,

$$e_{(x,y)} = \begin{cases} 1 & \text{in } (x, y) \text{ (coordinate)} \\ 0 & elsewhere \end{cases} \in A^{(M \times N)}$$

$\{e_{(x,y)} | (x,y) \in M \times N\}$ is a basis of $A^{(M\times N)}$.
L = submodule of $A^{(M\times N)}$ generally by the element

$$e_{(x_1, x_2, y)} - e_{(x_1,y)} - e_{(x_2,y)}$$

$$e_{(x,y_1+y_2)} - e_{(x,y_1)} - e_{(x,y_2)}$$

$x_i \in M$, $y_i \in N$.

$$e_{(ax,y)} - e_{(x,ay)},\ e_{(ax,y)} - ae_{(x,y)},\ a \in A$$

Prove that $\rho : M \times N \to A^{M\times N}/L$ is a tensor product of M, N over A.

□

For $x \in M$, $y \in N$, we note $x \otimes y = \rho(x,y)$.

Remarque 1.27. — *every of $M \otimes_A N$ can be written as $\sum_{i\ finite} x_i \otimes y_i$, $x_i \in M$, $y_i \in N$;*
— *$x \otimes y = x' \otimes y'$ does not imply a relation between x and x', y and y'.*

Example 1.28.

$$\mathbb{Z}/2\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}/3\mathbb{Z} = \{0\}$$

$x \otimes y = (3x - 2x) \otimes y = 3x \otimes y - 2x/6 \otimes y = x \otimes 3y - 0 \otimes y = x \otimes 0 - 0 \otimes y = f(x,0) - f(0,y) = 0 - 0 = 0$

Exercise 1.29. *Show that $\mathbb{Z}/m\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}/n\mathbb{Z} = \{0\}$ if m,n are coprime.*

Proposition 1.30. *A ring, M, N A-modules*

1. *$M \otimes_A A \simeq M$;*
2. *$M \otimes N \simeq N \otimes_A M$;*
3. *$(\oplus_i M_i) \otimes_A N \simeq \oplus_i (M_i \otimes_A N)$;*
4. *$L \otimes (M \otimes N) \simeq (L \otimes M) \otimes N$.*

Proof. 1. $M \otimes_A A \simeq M$ sending $x \otimes a = (ax) \otimes 1$ to ax.
Let $M \times A \to M$ sending (x,a) to xa. It is a bilinear map such that the following diagram is commutative:

$$\begin{tikzcd} M \times A \arrow[r] \arrow[d, "\rho"'] & M \\ M \otimes_A A \arrow[ur] & \end{tikzcd}$$

2. same kind of proof: we define $M \otimes_A N \simeq N \otimes_A M$ by mapping $x \otimes y$ to $y \otimes x$.
3. We define $(\oplus_i M_i \otimes_A N \simeq \oplus_i (M_i \otimes_A N)$ by mapping $(x_i)_i \otimes y \mapsto (x_i \otimes y)_i$.

□

Corollary 1.31. *If M is free over A with a basis $(e_\alpha)_\alpha$ then every elements g $M \otimes_A N$ can be written uniquely as $\sum_\alpha e_\alpha \otimes y_\alpha$, $y_\alpha \in N$,*

$$\begin{array}{lcl} (\oplus_i M_i) \otimes_A N & \simeq & \oplus_i (M_i \otimes_A N) \\ (x_i)_i \oplus y & \mapsto & (x_i \otimes y)_i \end{array}$$

Proof. use Proposition (c) $\square$

Tensor products of linear maps M, N, M', N' A-modules $u : M \to N'$, $v : M' \to N'$

$$\begin{tikzcd} M \otimes N \arrow[r, "u\otimes v"] & M' \otimes N' \\ M \times N \arrow[ur, "g"] & \end{tikzcd}$$

where g sends (x, y) to $u(x) \otimes v(x)$ is bilinear and $(u \otimes v)(x \oplus y) = u(x) \otimes v(y)$.

1.4 Base change or extension of scalars:

A ring, $\pi : A \to B$ ring homomorphism. If N is a B-module then N is a A-module. $a \in A$, $x \in N$, $x * a = \pi(a)x$.
If M is a A-module, $B \otimes_A M$ is a B-module, $b * (\sum_i b_i \otimes x_i) = \sum_i (bb_i) \otimes x_i$ $b_i \in B$, $x_i \in M$, $b \in B$.
Fix $b \in B$, we define the morphism $\cdot b : B \otimes M \to B \otimes M \to B \otimes M$ sending (c, x) to $(bc) \otimes x$

2 Complex of modules over A

Definition 2.1.

$$\ldots \longrightarrow M_0 \xrightarrow{f_0} M_1 \xrightarrow{f_1} M_2 \xrightarrow{f_2} \ldots \quad (*)$$

M_i A-module f_i linear maps.

1. *$(*)$ is a complex if $f_{i+1} \circ f_i = 0$ $\forall i$, that is $Im(f_i) \subset ker(f_{i+1})$.*

2. *A complex $(*)$ is exact if $Im(f_i) = ker(f_{i+1})$ $\forall i$ (exact sequence).*

Example 2.2. *The complex*

$$0 \longrightarrow M_0 \xrightarrow{f_0} M_1 \xrightarrow{f_1} M_2 \longrightarrow 0$$

1. *is exact at right if f_1 is surjective;*
2. *is exact at left if f_0 is injective;*
3. *is exact at the middle if $Im(f_0) = Ker(f_1)$.*

Exercise 2.3. *Let $0 \longrightarrow M_0 \xrightarrow{f_0} M_1 \xrightarrow{f_1} M_2 \longrightarrow 0$ be an exact sequence of A-modules. If M_0, M_2 are finitely generated, so is M.*

Proposition 2.4. *Let*

$$0 \longrightarrow M_0 \xrightarrow{f_0} M_1 \xrightarrow{f_1} M_2 \longrightarrow 0$$

be a short exact sequence. Let N be a A-module then, the complex

$$0 \longrightarrow M_0 \otimes_A N \xrightarrow{f_0} M_1 \otimes_A N \xrightarrow{f_1} M_2 \otimes_A N \longrightarrow 0$$

is exact (at right), that is $M_1 \otimes_A N \twoheadrightarrow M_2 \otimes_A N$.

Proof. Exercise. □

Corollary 2.5. *I ideal of A, M A-module then $M \otimes_A A/I \simeq M/IM$*

Proof. Exercise □

Remarque 2.6. *$I \otimes_A M \twoheadrightarrow IM$ (surjective but in general not injective.)*

Example 2.7. *$A = \mathbb{Z}$, $M = \mathbb{Z}/2\mathbb{Z}$, $I = 2\mathbb{Z}$, $IM = 2M = 0$.*
$I \simeq A$, $I \otimes_A M \simeq A \otimes_A M \simeq M \neq 0$.

Definition 2.8. *We say that M is a flat A-module if $\forall I$ ideal of A, the canonical map $I \otimes_A M \to IM$ is an isomorphism ($\Leftrightarrow I \otimes_A M \to M$ injective, sending $\alpha \mapsto \alpha x$.)*

Theorem 2.9. *M is flat $\Leftrightarrow$ for any injective morphism $N_1 \to N_2$ linear map of A-module then M is flat $\Leftrightarrow$ M is torsion free.*
Let M be a module on an integral domain A. M is torsion free, if $ax = 0$, $a \in A$ $\Rightarrow$ $a = 0$ or $x = 0$ that is equivalent to $\forall a \in A\backslash\{0\}$, $\cdot a : M \to M$ sending x to ax is injective.

Proof. Exercise. □

Exercise 2.10.
1. *Let A be a nonzero ring. Show that $A^m \simeq A^n$ then $m = n$.*
2. *Could you use the same proof to show that if $f : A^m \to A^n$ is surjective, then $m \geqslant n$?*
3. *Could you use the same proof to show that if $f : A^m \to A^n$ is injective, then $m \leqslant n$?*

2.1 Tensor product of algebras

Definition 2.11. *A is a ring, a A-algebra is a (commutative unitary) ring B endowed with a ring homomorphism $A \to B$.*

Example 2.12.
1. *$A[X_1, ..., X_n]$ is a A-algebra.*
2. *$A[X_1, ..., X_n]/I$ is a A-algebra/*
3. *Any ring is uniquely a $\mathbb{Z}$-algebra ($\forall$ B ring $\exists!$ ring homomorphism $\mathbb{Z} \to B$ sending k to $k \cdot 1_B$.*

Definition 2.13. *A finite generated A algebra is a A-algebra isomorphic to $A[x_1, ..., x_n]/I$*

B, C A-algebra, $B \otimes_A C$ exists as A-module and has a structure of A-algebra. We define the product as $(\sum b_i \otimes c_i)(\sum b'_j \otimes c'_j) = \sum b_i b'_j \otimes c_i c'_j$. It is well defined (independent of the representative) $A \to B \otimes_A C$ sending a to $a \otimes 1 = 1 \otimes a$ ring homomorphism.

Proposition 2.14. *Given B, C two algebra. For any A-algebra D, and ring homomorphism $\phi : B \to D$, $\psi : C \to D$, there exists a unique ring homomorphism $B \otimes_A C \to D$.*

$$\begin{tikzcd}
B \arrow[d, "i_B"] \arrow[dr, "\phi"] & \\
B \otimes_A C \arrow[r, "\theta"] & D \\
C \arrow[u, "i_C"] \arrow[ur, "\psi"] &
\end{tikzcd}$$

is commutative. Here, i_B sends b to $b \otimes 1$ and c to $1 \otimes c$.

Proof. Exercise. □

2.2 Nakayama lemma

Theorem 2.15. $(a, \mathfrak{m}_0)$ *a local ring (i.e. $\mathfrak{m}$ the unique maximal ideal of A. Let M be a finitely generated A-module such that $M = \mathfrak{m}_0 M$ then $M = 0$.*

Proof. Exercise. □

Proposition 2.16. *Let M be a A-module then M is flat if and only if for any B prime ideal of A, $M \otimes_A B$ is flat over B if and only if for any $\mathfrak{m}$ maximal ideal of A, $M \otimes_A A_{\mathfrak{m}}$ is flat over $A_{\mathfrak{m}}$*

Proof. Exercise. □

Exercise 2.17. *Prove that if A is a local ring, M and N are finitely generated A-modules, and $M \otimes_A N = 0$, then one of M or N is zero.*

Theorem 2.18. *Let $(A, \mathfrak{m})$ be a local ring. Let M be a finitely generated A-module then M is flat if and only if M is free.*

Proof. M free $\Rightarrow$ M flat in general even if M is not finitely generated. Suppose that M is flat $M \otimes_A A/\mathfrak{m} = M \otimes_A j \simeq M/\mathfrak{m}M \to k = A/\mathfrak{m}$ (k is a field the residue field of A.) is a vector space over k of finite dimension. If $x_1, ..., x_n \in M$ are such that $\bar{x_1}, ..., \bar{x_n} \in M \otimes_A k$ is a basis.
We want to prove $\{x_1, ..., x_n\}$ is a bases of M over A.

1. If $\{x_1, ..., x_n\}$ in M such that $\{\bar{x_1}, ..., \bar{x_n}\}$ genrates $M \otimes k$ implies $\{x_1, ..., x_n\}$ generates M.
2. If $\{x_1, ..., x_n\}$ in M such that $\{\bar{x_1}, ..., \bar{x_n}\}$ is free implies $\{x_1, ..., x_n\}$ is free.

LEFT IN EXERCISE □

Exercise 2.19. *If M and N are flat A-modules, then so is $M \otimes_A N$.*

3 Hilbert Nullstellensatz

Theorem 3.1. *(Hilbert Nullstellensatz) $A \to B$, B a A-algebra, $b \in B$ is integral over A if there is $a_0, ..., a_{n-1} \in A$ such that $a_0 + a_1 b + \cdots + a_{n-1}b^{n-1} + b^n = 0$:*

1. *$\{b \in B, b$ integral over $A\}$ subring of A;*
2. *B is finitely over A (definition is finitely generated as A-module) $\Leftrightarrow$ B is integral over A and finitely generated as A-algebra.*

k field

Lemma 3.2. *(Noether normalization lemma) If B is a finitely generated k-algebra then there exists a finite ring homomorphism $k[X_1, ..., X_n] \hookrightarrow B$.*

Theorem 3.3. *(Weak Hilbert Nullstellensatz) Let $\mathfrak{m}$ be a maximal ideal of $k[X_1, .., X_n]$ then $k \to k[X_1, ..., X_n]/\mathfrak{m}$ is a finite extension (if $k = \bar{f}$ algebraically closed, then $k[X_1, ..., X_n]/\mathfrak{m} \simeq k$ and $\mathfrak{m} = (X_1 - \alpha_1, ..., X_n - \alpha_n)$.*

Theorem 3.4. *(Strong Nullstellensatz) For any ideal of $k[X_1, .., X_n]$,*

$$\sqrt{I} = \bigcap_{\mathfrak{m} \text{ maximal ideal containing } I} \mathfrak{m}$$

Corollary 3.5. *Suppose that $k = \bar{k}$, let I be an ideal of $k[X_1, ..., X_n]$, let*

$$Z(I) = \{(\alpha_1, ..., \alpha_n) \in k^n | P(\alpha_1, \ldots, \alpha_n) = 0, \forall P \in I\}$$

Let $F \in k[X_1, ..., X_n]$ then $F(x) = 0$, $\forall x \in Z(I) \Leftrightarrow F \in \sqrt{I}$.

M, N A-modules $M' \subseteq M$, $N' \subseteq N$,

$$M/M' \otimes_A N/N' \simeq M \otimes_A N/(Im(M' \otimes N) + Im(M \otimes N'))$$

$$i : M' \to m,\ i_N = i \otimes Id_n : M' \otimes_A N \to M \otimes_A N$$

$$b : M/N' \times N/N' \to M \otimes_A N/(Im(M' \otimes N) + Im(M \otimes N')) \text{ bilinear sending } (\bar{x}, \bar{y}) \text{ to } \overline{x \otimes y}$$

$$\Rightarrow b : M/N' \otimes_A N/N' \to M \otimes_A N/(Im(M' \otimes N) + Im(M \otimes N')) \text{ sending } \bar{x} \otimes \bar{y} \text{ to } \overline{x \otimes y}$$

$$\text{Let } d : M \times N \to M/M' \otimes N/N' \text{ bilinear sending } (x, y) \text{ to } \bar{x} \otimes \bar{y}$$

$$\begin{tikzcd}
M \times N \arrow[r, "d"] \arrow[d, "\phi"] & M/M' \otimes N/N' \\
M \otimes_A N \arrow[ur] \arrow[d] & \\
M \otimes_A N/(Im(M' \otimes N) + Im(M \otimes N')) \arrow[uur, "\exists! d' \ linear"'] &
\end{tikzcd}$$

$Im(M' \otimes N) + Im(M \otimes N') \subset ker(\tilde{d})$, $d'(x' \otimes y + x \otimes y') = \bar{x}' \otimes \bar{y} + \bar{x} \otimes \bar{y}' = 0$.
We check that $\tilde{b}$ and $\tilde{d}$ are inverse one from another.

Exercise 3.6. *Explain how to deduce the Weak Nullstellensatz from the Strong Nullstellensatz.*